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BPS Monopoles in Moduli Space under SU(2) Gauge Potential

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Abstract Solving Dirac equation for a BPS monopole moving in the field of another BPS monopole in moduli space, it has been shown that spin momentum of the interacting monopole behaves as an extra energy source. The possibilities of splitting of the energy levels of the system have been explored.

Keywords BPS monopole · Moduli space · Gauge transformation · Non-relativistic approximation · Four-potential · Spin and orbital angular momentum operator

1 Introduction

The t' Hooft–Polyakov monopole [1, 2] is not an elementary particle like that of Dirac but a complicated extended object having a definite mass and finite size inside of which massive fields play a role in providing a smooth structure and outside of it they vanish rapidly leaving the field configuration identical to Abelian Dirac monopole. The t' Hooft–Polyakov monopole was only known numerically but there is simplified model introduced by Prasad and Sommerfield [3–8], which has an explicit stable monopole solutions. Such solutions satisfying Bogomol'nyi's condition [3, 4] are Bogomol'nyi–Prasad–Sommerfield (BPS) monopoles. These static monopoles in R^3-space have been extensively studies in recent years and it became clear that they have remarkable properties, which are best understood as a special case of self-duality equation in four-space for solutions independent of one of the variables. The hyperkahler property of the metric has been exploited for better mathematical understanding of BPS monopoles in moduli space by several authors [8, 9] and clear physical interpretation of coupling of Dirac operator with 't Hooft Polyakov monopole was given by Jackiw and Rebbi [10]. Later Callias [10] rigorously proved an index theorem for the Dirac operator in the background of a general BPS (multi-)monopole. Further studies of moduli space of monopoles with normalized bosonic and fermionic zero modes has been

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carried out [12, 13] for extended structure of non-Abelian monopoles and dyons in the same space.

In the present paper, we have solved the Dirac equation for a system of BPS monopole in moduli space under SU(2) gauge potential. The study of interaction of spin and orbital angular momentum has been undertaken and it is shown that extra-energy term may be interpreted as the energy of interaction of the spin moment of a BPS monopole with a vector field.

2 Extra Energy Operator and Dirac Matrices for Non-Abelian Monopole in SU(2) Gauge Theory

For BPS monopoles denoted by A_i $(i = 1, 2, 3)$, the Cartesian components of SU(2) gauge potential $\vec{A}$ will be treated on R^3 and by ϕ; a Higgs field in the adjoint representation of SU(2). Following the same notational conventions [7, 8] and writing su(2) for the Lie algebra of SU(2), we have four maps which are written as

$$A_i, \phi : R^3 \to \mathrm{SU}(2).$$

The following covariant derivative may be defined in terms of gauge potential A_i;

$$D_i = \partial_i + gA_i, \tag{2.1}$$

where g is the coupling parameter which has been taken as magnetic charge of BPS monopole, and the curvature is given by A_i

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j].$$

Gauge group SU(2) can be referred as the isospin group in which its basis can be written as:

$$t_a \quad (a = 1, 2, 3). \tag{2.1a}$$

This basis satisfies the commutation rule;

$$[t_a, t_b] = \varepsilon_{abc} t_c. \tag{2.2}$$

This equation can also be expressed in terms of the Pauli matrices τ_i which are given as:

$$t_a = (1/2i)\tau_a.$$

A boundary condition can be written by introducing a space A of pair (A, ϕ) as

$$\lim_{|\vec{x}| \to \infty} \|\phi(\vec{x})\| = 1, \tag{2.2a}$$

with the base point condition $\lim_{x_3 \to \infty} \phi(0, 0, x_3) = -t_3$.

The group G of gauge transformation is the space of maps $g : R^3 \to \mathrm{SU}(2)$ which satisfy the base point condition

$$\lim_{x_3 \to \infty} g(0, 0, x_3) = l_2. \tag{2.3}$$

The true configuration space is obtained as

$$\wp = A/G, \tag{2.4}$$

where G acts freely on A. As a result a non-Abelian magnetic field is introduced as

$$B_i = (1/2)\varepsilon_{ijk}F_{jk}. \tag{2.4a}$$

It defines a BPS monopole as a pair $(A, \phi) \in A$ which satisfies the Bogomol'nyi equation

$$B_i = D_i\phi. \tag{2.5}$$

Now introducing the notation X_k for the space of all k-monopoles, which is an infinite dimensional manifold on which the group G acts freely. Quenching X_k by this action we obtain the moduli space M_k, which is differentiable manifold of dimension $4k$. It has a natural Riemannian metric introduced from the Yang–Mills Higgs kinetic energy functional and it is explained [8] why M_k equipped with this metric is a hyperkahler manifold. M_l is a flat manifold of the form

$$M_l = R^3 \times S^l. \tag{2.6}$$

For a four component spinor, $\psi(t, \vec{x})$ can be written in the form of the fundamental representation of the SU(2) isospin group as

$$[-\{\Gamma^0 \otimes \partial_t + c\Gamma^\mu \otimes D_\mu\} + mc^2]\psi = 0, \tag{2.7}$$

where m is the mass of a BPS monopole while D_i is the covariant derivative given by (2.1) and $D_4 = \phi$. It is considered that SU(2) gauge potential A_μ, $\mu = 1, 2, 3, 4$ on $R^4 = R^3 \times R$, which is independent of X_4 and has self-dual curvature acts

$$F_{k\lambda} = (1/2)\varepsilon_{k\lambda\mu\nu}F_{\mu\nu}. \tag{2.8}$$

Identifying the fourth component of gauge potential with the Higgs field, (2.8) appear equivalent to the Bogomol'nyi equation. Five 4×4 complex matrices (Γ^0, Γ^4) can be obtained from the standard Dirac γ-matrices;

$$\Gamma^0 = \gamma^0, \qquad \Gamma^i = \gamma^i, \qquad \Gamma^4 = -i\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3. \tag{2.9}$$

$\psi(t, \vec{x})$ really transforms under a spinor representation of $SO(1, 4)$ but we can think of it as an $SO(1, 3)$ spinor by restricting to the Lorentz transformations in $SO(1, 3) \subset SO(1, 4)$ respecting the condition $X_4 = 0$ for a BPS monopole moving in an external field of another BPS monopole.

The Dirac equation (2.7) is modified to

$$[-\{\Gamma^0 \otimes \partial_t + c\Gamma^\mu \otimes (\partial_\mu + gA_\mu)\} + mc^2]\psi = 0. \tag{2.10}$$

On multiplying by Γ^4, and by using (2.9), this equation gives

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\begin{bmatrix} l_2 & 0 \\ 0 & -l_2 \end{bmatrix}\psi = \Bigg[-c\begin{bmatrix} 0 & \sigma_i \otimes D_i \\ -\sigma_i \otimes D_i & 0 \end{bmatrix} - \begin{bmatrix} l_2 \otimes \phi & 0 \\ 0 & -l_2 \otimes \phi \end{bmatrix} \\ + \begin{bmatrix} l_2 & 0 \\ 0 & -l_2 \end{bmatrix} mc^2\Bigg]\psi = 0. \end{aligned} \tag{2.11}$$

On replacing ψ by $\psi' = \psi^{\varepsilon^{-imc^2t/\hbar}}$ with $\psi' = \binom{\xi}{\eta}$, in terms of two-component functions, this equation may be written as

$$[i\hbar(\partial/\partial t) + l_2 \otimes \phi]\xi = -c\sigma_i \otimes (\partial_i + gA_i)\eta, \tag{2.12}$$

$$-[i\hbar(\partial/\partial t) + l_2 \otimes \phi + 2mc^2]\eta = c\sigma_i \otimes (\partial_i + gA_i)\xi. \tag{2.13}$$

In the first approximation, only the term $2mc^2\eta$ is retained on the left-hand side ode (2.13) and we have

$$\eta = -\left(\frac{1}{2mc}\right)\sigma_i \otimes (\partial_i + gA_i)\xi. \tag{2.14}$$

Which when substituted in (2.12), gives the following Pauli equations for BPS monopole in moduli space

$$i\hbar\frac{\partial \xi}{\partial t} = \left[\frac{1}{2m}(\vec{P}_i + gA_i)^2 - l_2 \otimes \phi - \frac{g\hbar}{2m}\sigma_i \otimes \operatorname{curl} A_i\right]\xi = \widehat{H}\xi, \tag{2.15}$$

where $\partial_i = \vec{P}_i$ is the momentum of BPS monopole. It shows that the following extra spin contribution in the energy is gained by BPS monopole while moving in the field of another BPS monopole;

$$E^i = -\mu_{g'}(\sigma_i \otimes \operatorname{curl} A_i), \tag{2.16}$$

where

$$\mu_{g'} = \frac{g\hbar}{2m}, \tag{2.17}$$

is defined as Bohr magneton for the system and

$$\mu_g = \mu_{g'}\sigma_i, \tag{2.18}$$

as spin moment of BPS monopole. This extra-energy may be interpreted as the energy of interaction of the spin moment of a BPS monopole with a vector field, resulting from the space rotation of four-potential $\{A_i^\mu\}$.

3 Contribution of Spin and Orbital Angular Momentum in the System of BPS Monopoles in Moduli Space

In order to analyze the contribution of both spin and angular momentum, we start by substituting $A_i = 0$ and $E = i\hbar\frac{\partial}{\partial t}$ in (2.12) and (2.13) gives

$$(E + l_2 \otimes \phi)\xi = -c\sigma_i \otimes \vec{P}_i\eta, \tag{3.1}$$

$$-(E - l_2 \otimes \phi + 2mc^2)\eta = c\sigma_i \otimes \vec{P}_i\xi. \tag{3.2}$$

These equations gives function η up to the first order in $(E - l_2 \otimes \phi)/2mc^2$. Substituting (3.2) into (3.1), we get

$$(E + l_2 \otimes \phi)\xi = \frac{1}{mc}(\sigma_i \otimes \vec{P}_i)\left[1 - \frac{(E - l_2 \otimes \phi)}{2mc^2}\right](\sigma_i \otimes \vec{P}_i)\xi, \tag{3.3}$$

which gives the following expression for Hamiltonian energy operator in the first approximation;

$$\widehat{H} = \left(\frac{1}{2m}\right)\left[1 - \frac{(E - l_2 \otimes \phi)}{2mc^2}\right]\vec{P}_i^2 - l_2 \otimes \phi - \left(\frac{i\hbar}{4m^2c^2}\right)[\vec{\nabla}(l_2 \otimes \phi) \otimes \vec{P}_i] + \left(\frac{\hbar}{4m^2c^2}\right)[\sigma_i \otimes \{\vec{\nabla}(l_2 \otimes \phi)\} \times \vec{P}_i]. \tag{3.4}$$

Now we will derive an expression for Hamiltonian in second order approximation, we use another function χ instead of ξ given as

$$\chi = \hat{u}\xi,$$

the normalization of which, up to second order, leads to the following value of factor $\hat{u}$,

$$\hat{u} \approx 1 - (\vec{P}_i^2/8m^2c^2).$$

We get the following relativistic expression for corresponding Hamiltonian up to terms of order v^2/c^2;

$$\begin{aligned}\widehat{H} &= \left[1 + \left(\frac{\vec{P}_i^2}{8m^2c^2}\right)\right]\widehat{H}\left[1 - \left(\frac{\vec{P}_i^2}{8m^2c^2}\right)\right] \\ &= \left[\left(\frac{\vec{P}_i^2}{2m}\right) - (l_2 \otimes \phi)\right] + \left[\left(\frac{\hbar^2}{8m^2c^2}\right)\vec{\nabla}^2(l_2 \otimes \phi)\right] - [(E - l_2 \otimes \phi)^2/2mc^2] \\ &\quad + \left(\frac{\hbar}{4m^2c^2}\right)[\sigma_i \otimes \{\vec{\nabla}(l_2 \otimes \phi)\} \times \vec{P}_i] \\ \widehat{H} &= \widehat{H}_0 + \widehat{H}_1 + \widehat{H}_2 + \widehat{H}_3.\end{aligned} \tag{3.5}$$

Here $\widehat{H}_0$ corresponds to the non-relativistic term of the Hamiltonian, while $\widehat{H}_1$ is the relativistic correction term to the Hamiltonian various parts of which arise due to different relativistic interaction. The quantity $\widehat{H}_1$ is called contact interaction operator, analogous to the term introduced by Darwin [14] for electronic case $\widehat{H}_2$ is the relativistic correction term due to the dependence of kinetic energy on momentum. Finally,

$$\widehat{H}_3 = \frac{\hbar[\sigma_i \otimes \{\vec{\nabla}(l_2 \otimes \phi)\} \times \vec{P}_i]}{4m^2c^2}, \tag{3.6}$$

is the spin-orbit interaction operator.

These relativistic corrections show the interaction of spin and angular momentum of a non-Abelian BPS monopole in SU(2) gauge theory that contributes to the energy operator besides the contribution of Higgs field which has been quoted in the beginning of the article.

4 Conclusion

Gauge potential is defined by (2.1) and the basis of the isospin group is given by (2.1a) whereas the commutation relation used for the same group is given by (2.2). A non-Abelian magnetic field is introduced as (2.4a) that defines a BPS monopole follows the Bogomol'nyi condition given by (2.5). Dirac equation has been written in terms of mass of a BPS monopole for the fundamental representation of SU(2) isospin group given as (2.11). Pauli equations for BPS monopole in moduli space that contains an extra term has been written as (2.15) and energy eigen value that contains Bohr's radius is given by (2.16). Hamiltonian energy operator in the first order approximation is given by (3.4) and the expression showing relativistic corrections for the same is given by (3.5). Individual term in the Hamiltonian representing spin-orbit interactions is given by (3.6). Hamiltonian of the system has been modified and it is taken in terms of Higgs potential in forthcoming papers [15, 16]. The problem has been treated in Abelian as well as in non-Abelian gauge theories and the effect due to moduli space approximation has been carried out.

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